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Linearization of Second Order Ordinary Differential Equations via Cartan's Equivalence Method

CHARLES GRISSOM

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205

GERARD THOMPSON

Department of Mathematics, University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Edinburgh EH9 3JZ, Great Britain

AND

GEORGE WILKENS

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205

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We use Cartan's equivalence method to study the differential invariants of a single second order ordinary differential equation relative to the pseudo-group of point transformations. As a result of the analysis a simple characterization is given of those second order equations which are linearizable by a point transformation.

I. INTRODUCTION

There has been some interest recently in the problem of determining when a given second order ordinary differential equation is equivalent via a point transformation to a linear equation [7]. Likewise Thompson [10] has investigated the linearizability problem for systems of autonomous second order differential equations and derived some surprisingly simple necessary and sufficient conditions for linearizability. The corresponding problem for non-autonomous systems is, however, much more complicated.

In this paper we present a solution to the linearization problem for the case of a single equation under point transformations, using É. Cartan's equivalence method [1]. Cartan considered the equivalence problem for

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second order equations under point transformations [2]; however, Cartan was more concerned with investigating the differential geometry of projective connections than studying second order differential equations for their own sake. Consequently, his treatment of the equivalence problem (Section 8 of [2]) is even harder to interpret than usual. In any case, with any work of Cartan one normally has to reconstruct it ab initio to fully understand and interpret it. Accordingly, we present here our version of the Cartan method, as imparted to us by our mutual advisor Robert Gardner, applied to second order equations. This will enable us not only to solve the linearization problem, but also prepares the way for future investigations on (Lie) infinitesimal symmetry groups of second order equations. Furthermore, our study adds to the growing number of problems in ordinary differential equations being analyzed by the Cartan method (compare [9, 5]).

We appreciate that in order to understand fully the Cartan equivalence method, at least as it is presented here, the reader will have to be fully conversant with modern differential geometry as presented in, for example, [8]. However, we trust that the theorem characterizing linearizable second order equations, presented in Section IV, will be accessible to a wide audience. In addition, in Section III we take some trouble to try to convey the essentially algorithmic nature of the equivalence method, which was clearly described for the first time in [4].

Finally, we make remarks concerning notation which are particularly intended to expedite the reading of Section III. First of all, we denote the exterior product of differential forms simply by juxtaposition, without a wedge product symbol. Second, we shall have occasion to write equations such as

$$\pi \equiv 0 \qquad (\operatorname{mod} \omega^1, \omega^2, \omega^3), \tag{1.1}$$

where π, ω^1, ω^2, ω^3 are 1-forms. By this we mean simply that π is a linear combination, possibly with functions coefficients, of ω^1, ω^2, ω^3. Third, we use the notation $J^1(\mathbf{R} \times \mathbf{R})$ to denote the bundle of 1-jets of locally defined functions from $\mathbf{R}$ to $\mathbf{R}$. $J^1(\mathbf{R} \times \mathbf{R})$ can be thought of as the three-dimensional space obtained by introducing the derivative as an independent variable. Fourth, we shall use frequently a result from exterior algebra known as Cartan's lemma, the most elementary form of which can be found in Sternberg [8, Chap. 1]. Fifth, and last, we commend to the reader several recent references which are germane to the discussion here, particularly [4, 5, 9, 11].

II. Preliminaries

We begin with some basic facts and definitions. We consider a single second order ordinary diffeential equation

$$y'' = F(x, y, y'). \tag{2.1}$$

(We assume of course that the equation is regular in the sense that the second order derivative may be solved for explicitly.) We view the equation as defining a line element field on $J^1(\mathbf{R} \times \mathbf{R})$ and a solution of it as curve on $J^1(\mathbf{R} \times \mathbf{R})$ which annihilates the two-dimensional Pfaffian module spanned by the 1-forms $\underline{\omega}^2 = dy' - F\,dx$, $\underline{\omega}^3 = dy - y'\,dx$.

We are concerned with the local equivalence of differential equations of the form (2.1) under the pseudogroup of point transformations. By a point transformation, we mean a local diffeomorphism of $\mathbf{R} \times \mathbf{R}$

$$X = X(x, y) \tag{2.2a}$$

$$Y = Y(x, y) \tag{2.2b}$$

which is extended in the natural way to a diffeomorphism of $J^1(\mathbf{R} \times \mathbf{R})$, by adjoining to (2.2a) and (2.2b) the equation

$$Y' = \frac{(\partial Y/\partial y)\, y' + \partial Y/\partial x}{(\partial X/\partial y)\, y' + \partial X/\partial x}. \tag{2.2c}$$

For future reference we also note that by a linear equation we mean one of the form

$$y'' = \lambda(x) + \mu(x)\, y + \nu(x)\, y' \tag{2.3}$$

and (2.1) is said to be linearizable if it can always be transformed locally by a point transformation to the form (2.3). (Of course we note that the solutions of (2.3) constitute technically an affine rather than a linear space unless λ is identically zero.)

An equation of the form (2.1) defines a G-structure on $J^1(\mathbf{R} \times \mathbf{R})$. By a G-structure on an n-dimensional manifold M, we mean a reduction of the coframe bundle FM to a principal bundle B_G with structure group G, a closed subgroup of $GL(n, \mathbf{R})$. Generally one may construct a (local) G-structure from an equivalence problem as follows. In an equivalence problem one is given open sets $U \subset \mathbf{R}^n$, $V \subset \mathbf{R}^n$ and coframes ${}'\underline{\omega} = (\underline{\omega}^1, ..., \underline{\omega}^n)$ and ${}'\underline{\Omega} = (\underline{\Omega}^1, ..., \underline{\Omega}^n)$ on U and V, respectively, and one seeks a diffeomorphism $\Phi\colon U \to V$ such that $\Phi^* \underline{\Omega} = \gamma_{VU} \underline{\omega}$, where γ_{VU} is a function with values in G. The local G-structures are then given by $U \times G$ and $V \times G$. It can be shown that the equivalences Φ are in one-to-one correspondence with diffeomorphisms $\Phi^{(1)}: U \times G \to V \times G$ such that $\Phi^{(1)*}\Omega = \omega$ where ω and Ω are the canonical $\mathbf{R}^n$-valued 1-forms on $U \times G$ and $V \times G$, respectively. (For more on G-structures see [4, 6, 8].)

In the case at hand, if we are a second order equation

$$Y'' = G(X, Y, Y') \tag{2.4}$$

then an equivalence is a map (2.2) which sends solution curves of (2.1) to solution curves of (2.4). If we choose $\underline{\omega}^1 = dx$ then ${}'\underline{\omega} = (\underline{\omega}^1, \underline{\omega}^2, \underline{\omega}^3)$ is a coframe "adapted" to (2.1) and we may choose analogously a coframe $\underline{\Omega}$ adapted to (2.4). Then a map Φ is an equivalence if and only if $\Phi^*\underline{\Omega} = \gamma_{VU}\underline{\omega}$, where γ_{VU} is of the form

$$\gamma_{VU} = \begin{pmatrix} A & 0 & B \\ 0 & C & E \\ 0 & 0 & AC \end{pmatrix}, \tag{2.5}$$

where AC is non-zero. The group G of our G-structure then consists of 3×3 matrices of the form of (2.5). The reader may easily verify that the canonical $\mathbf{R}^3$-valued 1-form ω on $B_G = U \times G$ is given by

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} A & 0 & B \\ 0 & C & E \\ 0 & 0 & AC \end{pmatrix} \begin{pmatrix} \underline{\omega}^1 \\ \underline{\omega}^2 \\ \underline{\omega}^3 \end{pmatrix}. \tag{2.6}$$

(The reader should compare [5, 9] where different problems in the context of second order ordinary differential equations are shown to lead to a G-structure.) In anticipation of the equivalence problem calculation to be performed in Section III, we note that the derivative of (2.6) (the structure equations) is given by

$$\begin{pmatrix} d\omega^1 \\ d\omega^2 \\ d\omega^3 \end{pmatrix} = \begin{pmatrix} dA/A & 0 & dB/AC - B\,dA/A^2C \\ 0 & dC/C & dE/AC - E\,dC/AC^2 \\ 0 & 0 & dA/A + dC/C \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$
$$+ \begin{pmatrix} (B/AC)\,\omega^1\omega^2 - (B^2/(AC^2)\,\omega^2\omega^3 + (BE/(AC)^2)\,\omega^3\omega^1 \\ ((E + F_{y'}C)/AC)\,\omega^1\omega^2 + (F_y/A^2 - F_{y'}E/A^2C - E^2/A^2C^2)\,\omega^1\omega^3 \\ + (F_{y'}B/A^2C + + BE/A^2C^2)\,\omega^2\omega^3 \\ \omega^1\omega^2 + (B/AC)\,\omega^2\omega^3 + (E/AC)\,\omega^3\omega^1 \end{pmatrix}. \tag{2.7}$$

III. Intrinsic Solution of the Equivalence Problem

We continue by applying the Cartan equivalence method to compute the differential invariants of equations of the form (2.1) under the pseudogroup of point transformations. The calculations are similar to those of [5, 9]. We remind the reader at the outset that there are two main elements in the Cartan method, namely, reduction and prolongation. Reduction consists of choosing more specialized coframes which capture finer geometric properties of the G-structure under consideration. Prolongation consists, roughly speaking, of computing "derivatives" of the G-structure and enables one to obtain differential invariants of higher and higher order.

One of the principal objectives of the Cartan method is to obtain, at some stage, an identity structure, that is a G-structure with G the trivial group. Such an identity-structure determines a coframing or parallelization of the manifold concerned and this particular coframe determines the equivalence class of the differential equation. In the case of an identity-structure the differential invariants depend only on finitely many derivatives of the data defining the structure and the problem of deciding whether two such G-structures are equivalent becomes, at least in principle, entirely deterministic (see [1, 4, 8]). We shall see below that the G-structures determined by second order equations lead to identity-structures.

Before proceeding to the calculation, we make another important conceptual point. We have described in Section II how a second order equation, with local presentation (2.1), leads to a G-structure. Furthermore, we recall that in [4] it was shown how the key idea in the Cartan method of analyzing a G-structure on an n-manifold M was to study the structure equations for the derivative $d\omega$ of the canonical $\mathbf{R}^n$-valued 1-form ω. Now ω is of course an invariant geometric object independent of coordinate considerations. As such, it is possible to apply the Cartan method in an invariant manner; that is, all the differential forms arising from the various reductions and prolongations are invariant. In that case one speaks of applying the method "intrinsically" (cf. [4, 9]).

However, in most cases, one would like to compute the "differential invariants" of the G-structure, to use a common phrase from Lie Theory. In other words, one would like invariants determined in terms of the function F in (2.1). Now the introduction of local coordinates enables one to trivialize the principal G-bundle over M associated to the G-structure and all subbundles and prolonged bundles obtained, respectively, by reduction and prolongation. The invariants embodied in the intrinsic form of the structure equations can then be realized in a concrete form. In fact the simplest procedure for computing the differential invariants is as follows. First of all, one performs the intrinsic calculation. This will generally lead to various cases because in the reduction process, one frequently has to make genericity assumptions, the validity of which can only be seen from a "parametric" calculation. (See [4, 8] for the significance of these genericity assumptions which amount to choosing G-orbits in the image of the structure function. In the equivalence problem we are considering here, however, we at no stage have to make such genericity assumptions.) Assuming then that eventially one obtains an identity-structure, one may then mimic the intrinsic calculation using the local data determining the G-structure and an initial choice of coframe.

Having made our preliminary comments we proceed next to use the Cartan equivalence method to study the local invariants of (2.1) under the pseudo-group of point transformations. Comparing with [9], we recall that in (2.6) and (2.7) ω is to be interpreted as the canonical $\mathbf{R}^3$-valued 1-form on the G-structure B_G. As such we must compute its exterior derivative $d\omega$ and absorb as much torsion (the terms quadratic in the ω^i's) as possible in a way which respects the Lie algebra g of G (see [4, 11] for a fuller explanation of the absorption technique). In fact we find that by defining

$$\alpha = \frac{dA}{A} - \frac{F_{y'}}{A}\omega^1 - \frac{B}{AC}\omega^2$$

$$\beta = \frac{dB}{AC} - \frac{B\,dA}{A^2C} - \frac{BE}{(AC)^2}\omega^1 + \frac{B^2}{(AC)^2}\omega^2$$

$$\gamma = \frac{dC}{C} + \frac{E + F_{y'}C}{AC}\omega^1 + \frac{2B}{AC}\omega^2$$

$$\varepsilon = \frac{dE}{AC} - \frac{E\,dC}{AC^2} + \frac{F_yC^2 - E(E + F_{y'}C)}{(AC)^2}\omega^1 + \frac{(E + F_{y'}CB)}{(AC)^2}\omega^2$$

we may rewrite (2.7) in the form

$$\begin{pmatrix} d\omega^1 \\ d\omega^2 \\ d\omega^3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & \varepsilon \\ 0 & 0 & \alpha+\gamma \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega^1\omega^2 \end{pmatrix}. \tag{3.1}$$

Here all products of 1-forms are exterior products and the 3×3 matrix on the right hand side of (3.1) is g-valued. (The fact that (3.1) resembles the structure equation for a linear connection explains our use of the term "torsion" for the terms quadratic in the ω^i's.)

Returning now to (3.1), since the torsion coefficients are constant, we cannot effect a group reduction (cf. [4, 8, 9]) and consequently, we must prolong. In order to prolong, we have to find the indeterminacy in the set of forms α, β, γ, and satisfying (3.1). If $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\varepsilon}$ are another set of forms satisfying (3.1) then we find by an application of Cartan's lemma that necessarily

$$\bar{\alpha} = \alpha + Q\omega^3 \tag{3.2a}$$

$$\bar{\beta} = \beta + Q\omega^1 + R\omega^3 \tag{3.2b}$$

$$\bar{\gamma} = \gamma + S\omega^3 \tag{3.2c}$$

$$\bar{\varepsilon} = \varepsilon + S\omega^2 + T\omega^3 \tag{3.2d}$$

for some functions Q, R, S, and T on B_G. The point of the prolongation procedure is now to use (3.2) to construct a G_0^1-structure with total space $B_{G_0^1}$ *over the space* B_G. (The reason for the notation G_0^1 for the new group will be explained presently.) In fact using (3.2) we can exhibit the group G_0^1 as follows in what is in effect the G_0^1-analogue of (2.6)

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \\ \bar{\varepsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 1 & 0 & 0 & 0 \\ Q & 0 & R & 0 & 1 & 0 & 0 \\ 0 & 0 & S & 0 & 0 & 1 & 0 \\ 0 & S & T & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \alpha \\ \beta \\ \gamma \\ \varepsilon \end{pmatrix}. \tag{3.3}$$

We must next find the analogue of $B_{G_0^1}$ of (3.1). Notice that since the base of our G_0^1-structure B_G is 7-dimensional, the torsion corresponding to each of $d\bar{\alpha}$, $d\bar{\beta}$, $d\bar{\gamma}$, and $d\bar{\varepsilon}$ contains, in principle, 21 terms. By differentiating (3.1) we can, however, see that most of those torsion terms are actually zero. Indeed we find that

$$0 = (d\bar{\alpha} + 2\bar{\varepsilon}\omega^1 + \bar{\beta}\omega^2)\,\omega^1 + (d\bar{\beta} - \bar{\beta}\bar{\gamma})\,\omega^3 \tag{3.4a}$$

$$0 = (d\bar{\gamma} - 2\bar{\beta}\omega^2 - \bar{\varepsilon}\omega^1)\,\omega^2 + (d\bar{\varepsilon} - \bar{\varepsilon}\bar{\alpha})\,\omega^3 \tag{3.4b}$$

$$0 = (d\bar{\alpha} + 2\bar{\varepsilon}\omega^1 + \bar{\beta}\omega^2)\,\omega^3 + (d\bar{\gamma} - 2\bar{\beta}\omega^2 - \bar{\varepsilon}\omega^1)\,\omega^3. \tag{3.4c}$$

Next applying Cartan's lemma several times to (3.4) we easily obtain

$$d\bar{\alpha} = -2\bar{\varepsilon}\omega^1 - \bar{\beta}\omega^2 + \theta\omega^3 + b\omega^1\omega^2 \tag{3.5a}$$

$$d\bar{\beta} = \bar{\beta}\bar{\gamma} + \theta\omega^1 + \rho\omega^3 \tag{3.5b}$$

$$d\bar{\gamma} = 2\bar{\beta}\omega^2 + \bar{\varepsilon}\omega^1 + \sigma\omega^3 - b\omega^1\omega^2 \tag{3.5c}$$

$$d\bar{\varepsilon} = -\bar{\alpha}\bar{\varepsilon} + \sigma\omega^3 + \tau\omega^3 \tag{3.5d}$$

for some real-valued function b and 1-forms θ, ρ, σ, and τ. Thus when we differentiate (3.3) and absorb compatibly with the Lie algebra g_0^1 of G_0^1 we obtain (dropping the bars for convenience)

$$d\begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{bmatrix} = \begin{bmatrix}0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&\theta&0&0&0&0\\ \theta&0&\rho&0&0&0&0\\ 0&0&\sigma&0&0&0&0\\ 0&\sigma&\tau&0&0&0&0\end{bmatrix}\begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{bmatrix} + \begin{bmatrix}\alpha\omega^1+\beta\omega^3\\ \gamma\omega^2+\varepsilon\omega^3\\ (\alpha+\gamma)\,\omega^3+\omega^1\omega^2\\ -b\omega^1\omega^2-2\varepsilon\omega^1-\beta\omega^2\\ \beta\gamma\\ b\omega^1\omega^2+2\beta\omega^2+\varepsilon\omega^1\\ \varepsilon\alpha\end{bmatrix}, \tag{3.6}$$

where the first matrix on the right hand side of (3.6) satisfies the Lie algebra relations of g_0^1.

The next stage in the equivalence method consists of trying to reduce the group G_0^1. The only (possibly) non-constant torsion coefficient in (3.6) is b and accordingly we compute the infinitesimal action of G_0^1 on this torsion component. (For more on the G-action on the torsion, compare [4, 8, 9] (Eq. (4) ff.)) Now differentiating the equation for $d\alpha$ in (3.6) and substituting in the result expressions for $d\omega^1$, $d\omega^2$, $d\omega^3$, $d\beta$, and $d\varepsilon$ from (3.6) we find

$$(db+2\theta-2\sigma)\,\omega^1\omega^2\omega^3\alpha\gamma=0 \tag{3.7}$$

and hence

$$db+2\theta-2\sigma\equiv 0 \qquad (\text{mod}\ \omega^1, \omega^2, \omega^3, \alpha,\ \gamma). \tag{3.8}$$

(The "mod" notation is explained in Section I.) Equation (3.8) means that b is acted on by translation and so we may achieve a reduction by setting b equal to zero. (For more on the reduction procedure see [4, 8, 9].)

The vanishing of b determines a principal subbundle of our G_0^1-structure, with structure group which we denote by G_1^1, in other words a G_1^1-structure. (The reason for the notation G_0^1 introduced above should now be evident: in general, G_q^p denotes the structure group corresponding to the qth reduction after p prolongations.) We can obtain the structure equations of the G_1^1-structure directly from (3.6) and (3.8) without the necessity of starting from the analogue of (2.5) or (3.3). We obtain, after absorption and considering equations similar to (3.5),

$$
d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&\theta&0&0&0&0\\\theta&0&\rho&0&0&0&0\\0&0&\theta&0&0&0&0\\0&\theta&\tau&0&0&0&0\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{pmatrix}+\begin{pmatrix}\alpha\omega^1+\beta\omega^3\\ \gamma\omega^2+\varepsilon\omega^3\\ (\alpha+\gamma)\,\omega^3+\omega^1\omega^2\\ -2\varepsilon\omega^1-\beta\omega^2\\ \beta\gamma\\ a\omega^1\omega^3+c\omega^2\omega^3+2\beta\omega^2+\varepsilon\omega^1\\ a\omega^1\omega^2+\varepsilon\alpha\end{pmatrix}. \tag{3.9}
$$

Again we consider the possibility of performing a group reduction and note that a and c in (3.9) are the only possible non-constant torsion coefficients. Accordingly, we determine the infinitesimal G_1^1-action on a and c. Proceeding much as we did to derive (3.7) and thence (3.8), we obtain from the derivatives of $d\alpha$, $d\varepsilon$ and $d\beta$, $d\gamma$, respectively,

$$
\begin{pmatrix}da\\ dc\end{pmatrix}\equiv\begin{pmatrix}3\tau\\ 3\rho\end{pmatrix}\qquad(\text{mod }\omega^1,\omega^2,\omega^3,\alpha,\gamma). \tag{3.10}
$$

An application of Cartan's lemma similar to that used in (3.4) reveals that on translating a and c to zero,

$$
\begin{pmatrix}\tau\\ \rho\end{pmatrix}\equiv\begin{pmatrix}0\\ 0\end{pmatrix}\qquad(\text{mod }\omega^1,\omega^2,\omega^3). \tag{3.11}
$$

The vanishing of a and c in (3.9) determines what, according to the notation we have established, is a G_2^1-structure. From (3.9) and (3.11) we easily derive the structure equations of the G_2^1-structure, after absorption, as

$$
d\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&\theta&0&0&0&0\\\theta&0&0&0&0&0&0\\0&0&\theta&0&0&0&0\\0&\theta&0&0&0&0&0\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \alpha\\ \beta\\ \gamma\\ \varepsilon\end{pmatrix}+\begin{pmatrix}\alpha\omega^1+\beta\omega^3\\ \gamma\omega^2+\varepsilon\omega^3\\ (\alpha+\gamma)\,\omega^3+\omega^1\omega^2\\ -2\varepsilon\omega^1-\beta\omega^2\\ I_1\omega^2\omega^3+\beta\gamma\\ 2\beta\omega^2+\varepsilon\omega^1\\ I_2\omega^1\omega^3+\varepsilon\alpha\end{pmatrix}. \tag{3.12}
$$

As usual, the next step is to see whether we can obtain a group reduction by investigating the G_2^1-action on the (possibly) non-constant torsion coefficients I_1 and I_2. From the derivatives of $d\beta$ and $d\varepsilon$ in (3.12) we find that

$$\begin{pmatrix} dI_1 \\ dI_2 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\text{mod } \omega^1, \omega^2, \omega^3, \alpha, \beta, \gamma, \varepsilon). \tag{3.13}$$

Equation (3.13) signifies that the functions I_1 and I_2 are "invariants"; by this we mean that I_1 and I_2 are (the pullbacks of) real-valued functions on the space B_G invariant under the G_2^1-action. In other words, if Φ is an automorphism of the G_2^1-structure (a diffeomorphism of $B_{G_2^1}$ preserving the canonical $\mathbf{R}^7$-valued 1-form; compare [4, 9]), then

$$\Phi^* I_1 = I_1 \tag{3.14a}$$

$$\Phi^* I_2 = I_2. \tag{3.14b}$$

Since I_1 and I_2 are invariants they cannot be used to obtain a group reduction and we must prolong to a G_0^2-structure over the bundle $B_{G_2^1}$. However, it is easy to check from (3.12) that the group G_0^2 is the trivial group $\{1\}$, or equivalently, that (3.12) determines θ uniquely. Thus the intrinsic part of the calculation is complete, save only for deriving an expression for the 2-form $d\theta$, which we do now as a final illustration of the use of Cartan's lemma.

From (3.12), differentiating either of the equations for $d\alpha$ or $d\gamma$ we obtain

$$(d\theta - \theta\alpha - \theta\gamma - \beta\varepsilon)\, \omega^3 = 0. \tag{3.15a}$$

Similarly, differentiating $d\beta$ and $d\varepsilon$, respectively, we obtain

$$(d\theta - \theta\alpha - \theta\gamma - \beta\varepsilon)\, \omega^1 + (dI_1 \omega^2 + 3I_1 \gamma\omega^2 + I_1 \alpha\omega^2)\, \omega^3 = 0 \tag{3.15b}$$

$$(d\theta - \theta\alpha - \theta\gamma - \beta\varepsilon)\, \omega^2 + (dI_2 \omega^1 + 3I_2 \alpha\omega^2 + I_1 \gamma\omega^2)\, \omega^3 = 0. \tag{3.15c}$$

From (3.15a) we have that

$$d\theta = \theta\alpha + \theta\gamma + \beta\varepsilon + \xi\omega^3 \tag{3.16}$$

for some 1-form ξ. Now from (3.15b) or (3.15c) we find easily that

$$d\theta = \theta\alpha + \theta\gamma + \beta\varepsilon + I_3 \omega^1\omega^3 + I_4 \omega^2\omega^3 \tag{3.17}$$

for some functions I_3 and I_4 on the G_0^2-structure. Thus we arrive at the following intrinsic form of the structure equations of the identity-structure:

$$d\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \alpha \\ \beta \\ \gamma \\ \varepsilon \\ \theta \end{bmatrix} = \begin{bmatrix} \alpha\omega^1 + \beta\omega^3 \\ \gamma\omega^2 + \varepsilon\omega^3 \\ (\alpha+\gamma)\,\omega^3 + \omega^1\omega^2 \\ -2\varepsilon\omega^1 - \beta\omega^2 + \theta\omega^3 \\ I_1\omega^2\omega^3 + \beta\gamma + \theta\omega^1 \\ 2\beta\omega^2 + \varepsilon\omega^1 + \theta\omega^3 \\ I_2\omega^2\omega^3 - \alpha\varepsilon + \theta\omega^2 \\ \theta\alpha + \theta\gamma + \beta\varepsilon + I_3\omega^1\omega^3 + I_4\omega^2\omega^3 \end{bmatrix}. \tag{3.18}$$

It is easy to see also from (3.18) that I_3 and I_4, in addition to I_1 and I_2, are invariant functions.

IV. Linearization and Parametric Solution of the Equivalence Problem

In the previous section we showed that the G-structures corresponding to second order equations are of order two; that is to say we obtain a identity-structure after two prolongations. In this section we first of all make some preliminary remarks about the structure equations (3.18) and then, in light of these, consider the linearization problem for second order equations.

Differentiating the equations for $d\beta$ and $d\varepsilon$, respectively, in (3.18) we obtain

$$(I_1\alpha + 3I_1\gamma + I_4\omega^1 + dI_1)\,\omega^2\omega^3 = 0 \tag{4.1}$$

$$(3I_2\alpha + I_2\gamma + I_3\omega^2 + dI_2)\,\omega^1\omega^3 = 0. \tag{4.2}$$

From (4.1) it is clear from the independence of the 1-forms considered that if I_1 is constant, then I_1 and I_4 are zero. Similarly from (4.2), if it is assumed that I_2 is constant, then one may conclude that actually I_2 and I_3 are zero.

We mentioned in Section III that in order to obtain the concrete form of the invariants in an equivalence problem, one must perform the parametric calculation in a way which mimics precisely the intrinsic one. Next we indicate briefly what this entails. Referring back to the first step in the intrinsic calculation, we note that (3.1) can be obtained by absorption from a knowledge of the group $G = G_0^0$ exhibited in (2.5). By contrast, the specific forms of α, β, γ, ε given prior to (3.1) are determined by the reference coframe $\underset{\sim}{\omega}^1$, $\underset{\sim}{\omega}^2$, $\underset{\sim}{\omega}^3$. The forms $\underset{\sim}{\omega}^1$, $\underset{\sim}{\omega}^2$, $\underset{\sim}{\omega}^3$ together with the specific form of α, β, γ, ε given prior to (3.1) (which could better now by written $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, $\underset{\sim}{\gamma}$, $\underset{\sim}{\varepsilon}$ to distinguish them from the coordinate-free forms appearing in (3.1)) furnish us with a moving coframe on the space B_G. Thus, starting from the reference coframe $(\underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \underset{\sim}{\omega}^3)$ on $J^1(\mathbf{R}\times\mathbf{R})$, we obtain a reference coframe $(\underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \underset{\sim}{\omega}^3, \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma}, \underset{\sim}{\varepsilon})$ on B_G and similarly a reference coframe on $B_{G_2^1}$. Geometrically the construction of those coframes derives from the fact that the coframe $(\underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \underset{\sim}{\omega}^3)$ yields a local trivialization of the bundle B_G and hence all of the bundles constructed over B_G by the process of reduction and prolongation. We see now, at least in principle, how we can obtain the parametric version of (3.18) and in particular expressions for the invariants I_1, I_2, I_3, I_4.

For future reference, we should like to note at this point that according to our calculations

$$I_1 = \frac{-1}{6AC^3} F_{y'y'y'y'} \tag{4.3}$$

$$I_2 = \frac{-1}{6A^3C}\left(\frac{d}{dx}(FF_{y'y'y'}) + \frac{d}{dx}F_{xy'y'} + y'\frac{d}{dx}F_{yy'y'} - 4\frac{d}{dx}F_{yy'}\right.$$
$$\left. - F_{y'}\frac{d}{dx}F_{y'y'} + FF_{yy'y'} + 6F_{yy} - 3F_yF_{y'y'} + 4F_{y'}F_{yy'}\right) \tag{4.4}$$

$$I_3 = -\left(\frac{1}{C}\frac{\partial}{\partial y'} + \frac{B}{AC}\right) I_2 \tag{4.5}$$

$$I_4 = -\left(\frac{1}{A}\frac{d}{dx} + \frac{2}{A}F_{y'} + \frac{E}{AC}\right) I_1, \tag{4.6}$$

where the subscripts on F indicate partial derivatives and the operator d/dx is defined by

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + F\frac{\partial}{\partial y'}.$$

Having obtained an identity structure with invariants I_1, I_2, I_3, I_4, the most natural question to ask is, what is the significance of the vanishing of the invariants? In fact the vanishing of I_1 and I_2 (and hence I_3 and I_4) are the necessary and sufficient conditions for (2.1) to be locally equivalent to

$$y'' = 0. \tag{4.7}$$

The necessity is clear and the sufficiency follows from the general fact that the equivalence class of an identity-structure is determined by its invariants and their covariant derivatives (see [4]). In this particular case, we can argue directly as follows. Given two equations (2.1) and (2.4) for which the associated invariants I_1 and I_2 vanish, consider the G_2^1 structures $B^1_{G_2^1}$ and $B^2_{G_2^1}$ with coframes ϕ_1 and ϕ_2, respectively, constructed as above. Then it is easy to check that on the product space $B^1_{G_2^1}$ the collection of forms $\phi_2 - \phi_1$ is a Frobenius system, and the leaves of the foliation are (locally) the graphs of maps $\Phi^{(1)}$: $B^1_{G_2^1} \to B^2_{G_2^1}$ which project to local equivalences of the base spaces. Thus any two such equations with vanishing invariants are locally equivalent; in particular they are all locally equivalent to $y'' = 0$.

The vanishing of I_1 implies that F in (2.1) satisfies

$$F = A + 3By' + 3C(y')^2 + D(y')^3 \tag{4.8}$$

for some functions A, B, C, D of x and y—a condition well known to be invariant under point transformations, but which is not obvious a priori (cf. [7]). Given (4.8), the vanishing of I_2 then gives the following two conditions:

$$2C_{xy} - B_{yy} - D_{xx} + AD_y + 2A_yD - 3B_xD - 3BD_x - 3B_yC + 6CC_x = 0 \tag{4.9}$$

$$B_{xy} - C_{xx} - A_xD - AD_x + 3A_yC + 3AC_y + 3BC_x - 6BB_y = 0. \tag{4.10}$$

It is somewhat remarkable that the vanishing of I_2 yields only the two conditions (4.9) and (4.10). In fact with G given by (4.8), I_2 is a degree four polynomial in y'; however, the terms of degree two, three, and four, respectively, vanish identically.

Let us now consider the linearization problem. In fact it is easy to verify that for F as given by the right hand side of (2.3), the invariants I_1 and I_2 both vanish. Thus we reach very naturally the conclusion that an equation (2.1) is linearizable if and only if it is equivalent to (4.5). Furthermore, the infinitesimal symmetry group of a linear or linearizable equation must be $SL(3, \mathbf{R})$ since it is the infinitesimal symmetry group of (4.8). We summarize our results more formally in the following

THEOREM. *The following (local) conditions are equivalent:*

(i) *Equation* (2.1) has $SL(3, \mathbf{R})$ *as its infinitesimal symmetry group,*

(ii) *Equation* (2.1) *is linearizable,*

(iii) *Equation* (2.1) *is locally equivalent to Eq.* (4.5),

(iv) *the* F *in* (2.1) *is a cubic polynomial in* y' *given by* (4.8), *say, for which, in addition,* (4.9) *and* (4.10) *are satisfied.*

V. Discussion

In this paper we have solved the problem of characterizing those second order ordinary differential equations which are linearizable by a point transformation using Cartan's equivalence method. The general equivalence problem for second order equations was studied by Tresse [12] using Lie methods. Cartan [2] also studied the equivalence problem but he was more concerned with investigating the differential geometry of projective connections. Accordingly, Cartan's account differs from his usual treatment of the equivalence problem. For example, Cartan gives the value of the invariant which in Section IV we denoted by I_1 as $-\frac{1}{6}F_{y'y'y'y'}$ and only gives the invariant I_2 in the restricted case where I_1 is zero. (There is also a typographical error in Cartan's formula for I_2 which he denotes by b: the y' in the formula for b given in paragraph 24 should multiply the first and not the second term in parentheses.)

In the Lie theory invariants such as I_1, I_2, I_3, I_4 are usually known as "semi"- or "relative" invariants. However, it seems to be difficult to ascribe a precise geometrical meaning to a semi-invariant within the Lie framework. As the Cartan method shows, the reason is that these invariants are properly invariants on the lifted G-structures. Thus while the vanishing of, for example, I_1 or I_2 yield conditions on the space $J^1(\mathbf{R}\times\mathbf{R})$, it is difficult to interpret I_1 or I_2 as determining geometric objects on $J^1(\mathbf{R}\times\mathbf{R})$.

Tresse's treatment of the equivalence problem for second order equations is, in its own terms, very complete including a discussion of infinitesimal symmetry groups and normal forms for equations with nontrivial groups of symmetries. However, it is generally agreed that Tresse's memoir is very difficult to read for a modern reader. Accordingly, we are currently engaged on the not inconsiderable task of interpreting and corroborating Tresse's results from the Cartan equivalence problem viewpoint, as well as answering some of the questions left unanswered by Tresse and Cartan.

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